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CONSTRUCTION OF AN ALGEBRAIC EQUATION WITH AN IRRATIONAL ROOT APPROXIMATELY EQUAL TO A GIVEN VALUE.[1]

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In teaching Horner's method it is often desirable to have an equation which shall exhibit a given peculiarity in the sequence of digits in the computed root. Thus it may be desired to illustrate the treatment of a zero, or the unreliability of the trial divisor for a large digit occurring early in the computation. The following method furnishes a means of finding an equation of any given degree above the first, with relatively small integral coefficients, in fact a large number of such equations, which shall have a root beginning with any given sequence of digits. No very definite meaning can be attached to the phrase "relatively small." The size of the coefficients will depend not only upon the number of digits given and the degree of the required equation, but also upon the closeness with which the given root happens to approximate to some simple algebraic irrationality. For example, if it were required to find an algebraic equation with a root beginning with 1.2679, it would not be especially obvious that this number is approximately equal to $3 - \sqrt{3}$, and hence satisfies the equation $x^2 - 6x + 6 = 0$.

The method can best be explained in connection with an example. As a certain interest attaches to an algebraic equation with a root approximately equal to π, let it be required to find an equation of the fourth degree having a root beginning with 3.14159. We have then the following values for the powers of this root:

$$x^4 = 97.40909,\ x^3 = 31.00628,\ x^2 = 9.86960,\ x = 3.14159,\ x^0 = 1.$$

We may now diminish the largest of these five numbers by subtracting from it a suitable multiple of one of the others, then diminish the largest remaining one in the same way, and so continue. The process is similar to Euclid's algorithm

[1] This note may be taken as an answer to algebra problem number 447, proposed in the December issue, though it was presented simultaneously with the problem.—EDITOR.

for the greatest common divisor, except that at each stage there is a choice of numbers whose multiples may be used. Thus

$$\begin{array}{rcrcl}
 & & A & = & x^4 - 3x^3 = 4.39025 \\
 & & B & = & x^3 - 3x^2 = 1.39747 \\
 & & C & = & x^2 - 3x = .44482 \\
A - 3B & = & D & = & x^4 - 6x^3 + 9x^2 = .19784 \\
 & & E & = & x - 3 = .14159 \\
B - 3C & = & F & = & x^3 - 6x^2 + 9x = .06301 \\
1 - 2C & = & G & = & -2x^2 + 6x + 1 = .11036 \\
C - 3E & = & H & = & x^2 - 6x + 9 = .02005 \\
D - 3F & = & I & = & x^4 - 9x^3 + 27x^2 - 27x = .00881 \\
E - G & = & J & = & 2x^2 - 5x - 4 = .03123 \\
2F - G & = & K & = & 2x^3 - 10x^2 + 12x - 1 = .01566 \\
F - 2J & = & L & = & x^3 - 10x^2 + 19x + 8 = .00055 \\
2K - J & = & M & = & 4x^3 - 22x^2 + 29x + 2 = .00009 \\
H - K & = & N & = & -2x^3 + 11x^2 - 18x + 10 = .00439 \\
2I - K & = & P & = & 2x^4 - 20x^3 + 64x^2 - 66x + 1 = .00196 \\
I - 2N & = & Q & = & x^4 - 5x^3 + 5x^2 + 9x - 20 = .00003 \\
N - 2P & = & R & = & -4x^4 + 38x^3 - 117x^2 + 114x + 8 = .00047 \\
4L - P & = & S & = & -2x^4 + 24x^3 - 104x^2 + 142x + 31 = .00024 \\
L - R & = & T & = & 4x^4 - 37x^3 + 107x^2 - 95x = .00008 \\
2S - R & = & U & = & 10x^3 - 91x^2 + 170x + 54 = .00001 \\
3T - S & = & V & = & 14x^4 - 135x^3 + 425x^2 - 427x - 31 = 0
\end{array}$$

Here then is an equation with a root beginning 3.14159. If we desire only one such equation we need go no further. In fact we might have stopped after finding Q, and have written at once $3Q - M = 0$. We may, however, go on to get other equations. Thus

$$\begin{array}{rcrcl}
3Q - M & = & W & = & 3x^4 - 19x^3 + 37x^2 - 2x - 62 = 0 \\
3Q - T & = & X & = & -x^4 + 22x^3 - 92x^2 + 122x - 60 = .00001 \\
Q - 3X & = & Y & = & 4x^4 - 71x^3 + 281x^2 - 357x + 160 = 0 \\
U - X & = & Z & = & x^4 - 12x^3 + x^2 + 48x + 114 = 0
\end{array}$$

V, W, Y, and Z, are four linearly independent polynomials, each having one root approximately equal to 3.14159. Therefore, any linear combination of these will also have such a root. We may thus get a large number of equations satisfying the conditions of the problem. Of these there may be some having coefficients smaller than those first found. To obtain these we may try to diminish the coefficients in the same way that we diminished the values of the polynomials, that is by getting rid of the largest, and then the next largest, and so on. Thus

$$\begin{array}{rcrcl}
V - Y & = & A_1 & = & 10x^4 - 64x^3 + 144x^2 - 70x - 191 = 0 \\
A_1 - Y & = & B_1 & = & 6x^4 + 7x^3 - 137x^2 + 287x - 351 = 0
\end{array}$$

These are improvements upon V and Y respectively. But it soon becomes difficult to diminish one coefficient without increasing another. The four following independent equations can be obtained after a little manipulation:

$$\begin{array}{rcrcl}
-64W + 6Z + 19A_1 + 3B_1 & = & C_1 & = & 22x^4 - 51x^3 - 37x^2 - 53x - 30 = 0 \\
W + Z & = & D_1 & = & 4x^4 - 31x^3 + 38x^2 + 46x + 52 = 0 \\
-21W + 2Z + 6A_1 + B_1 & = & E_1 & = & 5x^4 - 2x^3 - 48x^2 + 5x + 33 = 0 \\
-20W + 2Z + 6A_1 + B_1 & = & F_1 & = & 8x^4 - 21x^3 - 11x^2 + 3x - 29 = 0
\end{array}$$

The last of these not only has remarkably small coefficients but also has a root more nearly equal to π than we had any right to expect, namely $x = 3.1415925$.

The last four equations may be combined to satisfy various further conditions. For example, $F_1 - 2D_1$ gives us the third degree equation

$$41x^3 - 87x^2 - 89x - 133 = 0.$$

Or we may obtain an equation making a still closer approximation. Thus, $3D_1 - F_1$ gives the equation

$$4x^4 - 72x^3 + 125x^2 + 135x + 185 = 0,$$

which has a root equal to 3.1415926557, the value of π being 3.1415926536. Or various other conditions might be imposed.

For the degree of accuracy used here the computation is rather laborious. If, however, only a single third degree equation is desired and not more than four or five significant figures in the root, the work is not long, especially if the powers of x at the beginning are found by logarithms.

NAPIER'S LOGARITHMIC CONCEPT: A REPLY.

By FLORIAN CAJORI.

In the *Mathematical Gazette* of May, 1915, page 78, Professor H. S. Carslaw quotes the following passage from my article: "A History of the Exponential and Logarithmic Concepts," in the AMERICAN MATHEMATICAL MONTHLY of January, 1913, page 7:

Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{cccccc} v, & v\left(1-\frac{1}{v}\right), & v\left(1-\frac{1}{v}\right)^2, & \cdots, & v\left(1-\frac{1}{v}\right)^n, & \cdots \\ 0, & 1, & 2, & \cdots, & n, & \cdots. \end{array}$$

The numbers in the upper series represent successive values of the *sines*; the numbers in the lower series stand for the corresponding logarithms. Thus $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log [10^7(1 - 10^{-7})^n] = n$, where $n = 0, 1, 2, \cdots$.

Professor Carslaw says: "This statement is incorrect. In Napier's Tables the logarithm of $(10^7 - 1)$ is not 1. It lies between 1 and 1.0000001, and he takes it as the mean between these two numbers, namely 1.00000005."

In reply to this I desire to make the following remarks: (1) In my article I did not explain at all Napier's *computation*; I aimed to explain his logarithmic *concept*. Napier's *theory* rests on the establishment of a one-to-one correspondence between the terms of a geometric series and the terms of an arithmetic series.[1] But, *it is not possible to write down two such series which represent exactly the numbers arising in Napier's computations.* Professor Carslaw himself admits that, in Napier's computations, "the numbers are not exactly in geometrical

[1] See Napier's *Constructio* (Macdonald's edition), page 19.